Lorentz - Kovarianz der Dirac - Gleichung

passive Trafo $I \to I'$

$x' = \Lambda x + a$

$\psi'(x') = S(\Lambda)\ \psi(x)$

$0 = (-i\gamma^\mu \partial_\mu + m)\psi(x) \iff (-i\gamma^\mu \partial'_\mu + m)\psi'(x') = 0$

infinitesimale Trafo $\quad \Lambda^\mu{}_\nu = g^\mu{}_\nu + \Delta\omega^\mu{}_\nu$

$\Rightarrow \quad \Delta\omega^{\nu\mu} = -\Delta\omega^{\mu\nu} \qquad$ 6 Parameter

a) 3 Raumdrehungen

$$\Delta\omega^1{}_2 = g^{22}\Delta\omega^1{}_2 = -\Delta\omega^{12} = \Delta\omega^{21} = -\Delta\omega^2{}_1 = \Delta\varphi$$

$$\Rightarrow \Lambda = 1\!\!1 + \begin{pmatrix} 0 & & & \\ & 0 & \Delta\varphi & 0 \\ 0 & -\Delta\varphi & & \\ 0 & 0 & & 0 \end{pmatrix} \Leftarrow \lim_{\Delta\varphi \to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Delta\varphi & \sin\Delta\varphi & 0 \\ 0 & -\sin\Delta\varphi & \cos\Delta\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Drehung um z-Achse mit Winkel $\Delta\varphi$

endliche Drehung um $\varphi = N \cdot \Delta\varphi$

$$\Lambda = \lim_{N\to\infty} \underbrace{\left(1\!\!1 + \frac{\varphi}{N}\begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 0 \end{pmatrix}\right)^N}_{N \text{ mal drehen}} = \exp\left(\varphi \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b) Boost (3 Richtungen)

z.B. $\quad \Delta\omega^0{}_1 = -\Delta\omega^{01} = +\Delta\omega^{10} = \Delta\omega^1{}_0 = -\Delta\eta$

$\left(\beta = \frac{v}{c}\ ,\ \gamma = \frac{1}{\sqrt{1-\beta^2}}\ ;\ \tanh\eta = \beta\right)$

$\Delta\beta \approx \Delta\eta$
für kleine Werte

$$\Rightarrow \Lambda = 1\!\!1 + \begin{pmatrix} 0 & -\Delta\eta & 0 & 0 \\ -\Delta\eta & 0 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & 0 \end{pmatrix} =$$

$$\lim_{\eta\to 0}\begin{pmatrix}\cosh\eta & -\sinh\eta & & 0\\ -\sinh\eta & \cosh\eta & & \\ & 0 & 1 & 0\\ & & 0 & 1\end{pmatrix}$$ Boost in x-Richtung

$$\Lambda = \lim_{N\to\infty}\left(1\!\!1 + \frac{\eta}{N}\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & \\ 0 & & 0\end{pmatrix}\right)^N = \exp\left(\eta\begin{pmatrix}0 & -1 & \\ -1 & 0 & 0\\ 0 & & 0\end{pmatrix}\right)$$

$$= \begin{pmatrix}\cosh\eta & -\sinh\eta & 0 & \\ -\sinh\eta & \cosh\eta & & \\ 0 & & 1 & 0\\ & & 0 & 1\end{pmatrix}$$

Ansatz infinit $S = 1\!\!1 + T$

$\longrightarrow \quad S^{-1} = 1\!\!1 - T$

$\det S = 1 \longrightarrow \text{Spur } T = 0$

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$$

$$(1\!\!1 - T)\gamma^\mu(1\!\!1 + T) = \gamma^\mu + \gamma^\mu T - T\gamma^\mu$$

$$= (g^\mu{}_\nu + \Delta\omega^\mu{}_\nu)\gamma^\nu$$

$$= \gamma^\mu + \Delta\omega^\mu{}_\nu\gamma^\nu$$

Rate, rate .... Lösung ist $T = \frac{1}{8}\Delta\omega^{\mu\nu}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

$$= -\frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}$$

mit $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] = -\sigma_{\nu\mu}$

wieder zu

a) Drehung um z-Achse: R_3

$$T_{R3}(\Delta\varphi) = \frac{i}{2}\Delta\varphi\,\sigma_{12}$$

$$\sigma_{12} = \frac{i}{2}[\gamma_1, \gamma_2]$$

$$= i\gamma_1\gamma_2$$

$$= i\begin{pmatrix}0 & \sigma^1\\ -\sigma^1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix} = \begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}$$

$$\Rightarrow \quad S_{R_3}(R_3) = \mathbb{1} + \frac{i}{2}\Delta\varphi \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

endliche Drehung um $\varphi = N\Delta\varphi$

$$S(\varphi) = \lim_{N\to\infty} \left(1 + \frac{i\varphi}{2N}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\right)^N = \exp\left(\frac{i\varphi}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\right)$$

$$= \cos\frac{\varphi}{2}\,\mathbb{1} + i\sin\frac{\varphi}{2}\,\sigma_{12}$$

$\Rightarrow$ Drehung um $\varphi = 2\pi$ liefert $S_{R_3}(2\pi) = -1$

erst um $\varphi = 4\pi$ liefert $S_{R_3}(4\pi) = 1$

b) Boost in x^1-Richtung : L_1

$$\sigma_{L_1}(\Delta\eta) = -\frac{i}{2}\Delta\eta\,\sigma_{01} = \frac{1}{2}\Delta\eta\,\alpha_1$$

$$\sigma_{01} = i\gamma_0\gamma_1 = i\beta\beta\alpha_1 = i\alpha_1 \qquad \alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad S_{L_1}(\Delta\eta) = \mathbb{1} + \frac{1}{2}\Delta\eta\,\alpha_1$$

endliche Boost

$$S_{L_1}(\eta) = \lim_{N\to\infty}\left(\mathbb{1} + \frac{\eta}{2N}\alpha_1\right)^N = \exp\left(\frac{\eta}{2}\alpha_1\right) = \mathbb{1}\cosh\frac{\eta}{2} + \alpha_1\sinh\frac{\eta}{2}$$

$$(\alpha_1)^2 = \mathbb{1}$$

Boost in bel. Richtung $\vec{v}$

$$S(\vec{v}) = \mathbb{1}\cosh\frac{\eta}{2} + \begin{pmatrix} 0 & -\frac{\vec{v}\vec{\sigma}}{v} \\ -\frac{\vec{v}\vec{\sigma}}{v} & 0 \end{pmatrix}\sinh\frac{\eta}{2}$$

$\beta = \frac{v}{c}$, $v = |\vec{v}|$

$\tanh\eta = \beta$

mit $\tanh\frac{\eta}{2} = \dfrac{\tanh\eta}{1+\sqrt{1-\tanh^2\eta}} = \dfrac{\beta}{1+\sqrt{1-\beta^2}}$

$$\vec{v} = \frac{\partial E}{\partial \vec{p}} = \frac{\vec{p}\, c^2}{E}$$

$$E = \sqrt{p^2 c^2 + m^2 c^4}$$

$$\Longleftrightarrow \quad \vec{p} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}\, \vec{v}$$

$$1 + \sqrt{1 - \beta^2} = 1 + \sqrt{1 - \frac{p^2 c^2}{E^2}} = \frac{|E| + mc^2}{|E|}$$

$$\Rightarrow \tanh \frac{\eta}{2} = \frac{pc}{|E| + mc^2} = \frac{\sqrt{E^2 - m^2 c^4}}{|E| + mc^2} = \sqrt{\frac{|E| - mc^2}{|E| + mc^2}}$$

$$\Rightarrow \cosh \frac{\eta}{2} = \frac{\sqrt{|E| + mc^2}}{\sqrt{2mc^2}} \;; \quad \sinh \frac{\eta}{2} = \frac{\sqrt{|E| + mc^2}}{\sqrt{2mc^2}} \frac{pc}{|E| + mc^2}$$

$$S(\vec{p}) = \sqrt{\frac{|E| + mc^2}{2mc^2}} \left[\mathbb{1} - \begin{pmatrix} 0 & \frac{\vec{p}\vec{\sigma}}{p} \\ \frac{\vec{p}\vec{\sigma}}{p} & 0 \end{pmatrix} \frac{pc}{|E| + mc^2} \right]$$

$$\frac{\vec{p}\vec{\sigma}}{p} = \frac{1}{p} \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & p_z \end{pmatrix} \qquad \text{def. } p_\pm = p_x \pm i p_y$$

$$S(\vec{p}) = \sqrt{\frac{|E| + mc^2}{2mc^2}} \begin{pmatrix} 1 & 0 & \frac{p_z c}{|E| + mc^2} & \frac{p_- c}{|E| + mc^2} \\ 0 & 1 & \frac{p_+ c}{|E| + mc^2} & \frac{-p_z c}{|E| + mc^2} \\ \frac{p_z c}{\cdots} & \frac{p_- c}{\cdots} & 1 & 0 \\ \frac{p_+ c}{\cdots} & \frac{-p_z c}{\cdots} & 0 & 1 \end{pmatrix}$$

$; \; -\vec{p}$

c) **Raumspiegelung**

$$\Lambda = P = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$S^{-1} \gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu = \sum_{\nu=1}^{4} g^{\mu\nu} \gamma^\nu = g^{\mu\mu} \gamma^\mu$$

(ohne summation über μ)

erfüllt durch

$S = e^{i\chi} \gamma^0$ mit bel. χ z.B. $\chi = 0$

$$\Rightarrow S^{-1} \gamma^\mu S = \gamma^0 \gamma^\mu \gamma^0 = \begin{cases} \gamma^{0\,3} = \gamma^0 & : \mu = 0 \\ -\gamma^{0\,2} \gamma^\mu = -\gamma^\mu & : \mu \neq 0 \end{cases}$$

Raumspiegelung Op $\mathcal{P}$

$$\psi(x) \xrightarrow{\mathcal{P}} \psi'(x') = \psi'(\vec{r}', t) = S\,\psi(x) = e^{i\chi} \gamma^0 \psi(x)$$

$$= e^{i\chi} \gamma^0 \psi(-\vec{r}', t)$$

$\mathcal{P} = e^{i\chi} \gamma^0 P_{orb}$ $\quad P_{orb}$ macht $\vec{r} \longrightarrow -\vec{r}$