Lorentz-Kovarianz der Diracgleichung

- passive Trafos $I \to I'$: $\boxed{x' = \Lambda x + a}$

$$\boxed{\psi'(x') = S(\Lambda)\,\psi(x)}$$

$$(-i\gamma^\mu \partial_\mu + m)\psi(x) \Longleftrightarrow (-i\gamma^\mu \partial_\mu' + m)\psi'(x)$$

$$\Rightarrow S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu \gamma^\mu$$

- infinitesimale Trafo: $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \Delta\omega^\mu{}_\nu$

$$\Rightarrow \boxed{\Delta\omega^{\nu\mu} = -\Delta\omega^{\mu\nu} \text{ (6 Parameter)}}$$

a) 3 Raumdrehungen z.B. Drehung um z-Achse mit $\Delta\varphi$

$$\Rightarrow \Lambda = \mathbb{1} + \begin{pmatrix} 0&0&0&0\\ 0&0&\Delta\varphi&0\\ 0&-\Delta\varphi&0&0\\ 0&0&0&0 \end{pmatrix} = \lim_{\Delta\varphi\to 0} \begin{pmatrix} 1&0&0&0\\ 0&\cos\Delta\varphi&\sin\Delta\varphi&0\\ 0&-\sin\Delta\varphi&\cos\Delta\varphi&0\\ 0&0&0&1 \end{pmatrix}$$

z.B. $\Delta\omega^1{}_2 = -\Delta\omega^{12} = \Delta\omega^{21} = -\Delta\omega^2{}_1 = \Delta\varphi$

- endliche Drehung um $\varphi = N\cdot\Delta\varphi$

$$\Lambda = \lim_{N\to\infty}\left(\mathbb{1} + \frac{\varphi}{N}\begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&-1&0&0\\ 0&0&0&0 \end{pmatrix}\right) = \exp\left[\varphi\begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&-1&0&0\\ 0&0&0&0 \end{pmatrix}\right]$$

(Übung)

$$\begin{pmatrix} 1&0&0&0\\ 0&\cos\varphi&\sin\varphi&0\\ 0&-\sin\varphi&\cos\varphi&0\\ 0&0&0&1 \end{pmatrix}$$

b) und 3 Boost (3 Richtungen) z.B. $\Delta\omega^0{}_1 = -\Delta\omega^{01} = \Delta\omega^{10} = \Delta\omega^1{}_0 = -\Delta\eta$

$$\boxed{\beta = \frac{v}{c}\ ;\ \gamma = \frac{1}{\sqrt{1-\beta^2}}\ ;\ \tanh\eta = \beta}$$

für kleine Argumente: $\Delta\beta = \Delta\eta$

- Boost in x-Richtung

$$\Rightarrow \Lambda = \mathbb{1} + \begin{pmatrix} 0&-\Delta\eta&0&0\\ -\Delta\eta&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} = \lim_{\Delta\eta\to 0}\begin{pmatrix} \cosh\Delta\eta&-\sinh\Delta\eta&0&0\\ -\sinh\Delta\eta&\cosh\Delta\eta&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

- endliches v oder $\eta = N\Delta\eta$

$$\Lambda = \lim_{N\to\infty}\left(\mathbb{1} + \frac{\eta}{N}\begin{pmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}\right)^N = \exp\Bigg[\eta\underbrace{\begin{pmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}}_{=I}\Bigg]$$

$$= \begin{pmatrix} \cosh\Delta\eta&-\sinh\Delta\eta&0&0\\ -\sinh\Delta\eta&\cosh\Delta\eta&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

$$I^2 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},\quad I^3 = I$$

Ansatz infinitesimal $S = \mathbb{1} + \tau \Rightarrow S^{-1} = \mathbb{1} - \tau$

$\det S = 1 \Longrightarrow \mathrm{Spur}(\tau) = 0$

$$\boxed{S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu}$$

$$(\mathbb{1}-\tau)\gamma^\mu(\mathbb{1}+\tau) = \gamma^\mu + \gamma^\mu\tau - \tau\gamma^\mu = (g^\mu{}_\nu + \Delta\omega^\mu{}_\nu)\gamma^\nu$$
$$= \gamma^\mu + \Delta\omega^\mu{}_\nu\gamma^\nu$$

Lösung: $\underline{\tau} = \frac{1}{8}\Delta\omega^{\mu\nu}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

$$\boxed{= -\frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}}$$ mit $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$

a) Drehung um z-Achse: R_3

$\Rightarrow \tau_{R_3}(\Delta\varphi) = \frac{i}{2}\Delta\varphi\,\sigma_{12}$ $\qquad \sigma_{12} = \frac{i}{2}[\gamma_1, \gamma_2] = i\gamma_1\gamma_2 = i\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0\end{pmatrix} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}$

$\Rightarrow S_{R_3}(\Delta\varphi) = \mathbb{1} + \frac{i}{2}\Delta\varphi\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}$

- endliche Drehung um z-Achse um $\varphi = N\cdot\Delta\varphi$

$$S(\varphi) = \lim_{N\to\infty}\left(\mathbb{1} + \frac{i\varphi}{2N}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}\right)^N = \exp\Big[i\frac{\varphi}{2}\underbrace{\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}}_{I}\Big] \qquad I^2 = \mathbb{1}$$
$$= \cos\left(\frac{\varphi}{2}\right)\mathbb{1} + i\sin\left(\frac{\varphi}{2}\right)\sigma_{12}$$

Drehung um $\varphi = 2\pi$ liefert $S_{R_3}(2\pi) = -1$

erst um $\varphi = 4\pi$ liefert $S_{R_3}(4\pi) = +1$

b) Boost in x^1-Richtung: L_1 $\qquad \sigma_{01} = \frac{i}{2}[\gamma_0, \gamma_1] = i\gamma_0\gamma_1 = i\beta\beta\alpha_1 = i\alpha_1$

$\tau_{L_1}(\Delta\eta) = -\frac{i}{2}\Delta\eta\,\sigma_{01} = \frac{1}{2}\Delta\eta\,\alpha_1$ $\qquad \alpha_1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0\end{pmatrix}$

$\Rightarrow S_{L_1}(\Delta\eta) = \mathbb{1} + \frac{1}{2}\Delta\eta\,\alpha_1$

- endlicher Boost $S_{L_1}(\eta) = \lim_{N\to\infty}\left(\mathbb{1} + \frac{\eta}{2N}\alpha_1\right)^N = \exp\left[\frac{\eta}{2}\alpha_1\right]$, $\alpha_1^2 = \mathbb{1}$

$$= \mathbb{1}\cosh\left(\frac{\eta}{2}\right) + \alpha_1\sinh\left(\frac{\eta}{2}\right)$$

Boost in beliebige Richtung $\vec{v}$ $\quad (\beta = \frac{v}{c};\ v = |\vec{v}|;\ \tanh\eta = \beta)$

$$S(\vec{v}) = \mathbb{1}\cosh\left(\frac{\eta}{2}\right) + \begin{pmatrix} 0 & -\frac{\vec{v}\cdot\vec{\sigma}}{v} \\ -\frac{\vec{v}\vec{\sigma}}{v} & 0\end{pmatrix}\sinh\left(\frac{\eta}{2}\right)$$

Umformungen:

$$\boxed{\tanh\frac{\eta}{2}} = \frac{\tanh\eta}{1 + \sqrt{1 - \tanh^2\eta}} \boxed{= \frac{\beta}{1 + \sqrt{1-\beta^2}}}$$

$$1 + \sqrt{1-\beta^2} = 1 + \sqrt{1 - \frac{p^2c^2}{E^2}} = \frac{|E| + mc^2}{|E|}$$

$$\Rightarrow \tanh\frac{\eta}{2} = \frac{pc}{|E| + mc^2} = \frac{\sqrt{E^2 - m^2c^4}}{|E| + mc^2}$$

$E = \sqrt{p^2c^2 + m^2c^4}$

$\vec{v} = \frac{\partial E}{\partial\vec{p}} = \frac{\vec{p}\cdot c^2}{E}$

$$\boxed{\vec{p} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}\vec{v}}$$

$$\boxed{\tanh\frac{\eta}{2} = \sqrt{\frac{|E|-mc^2}{|E|+mc^2}}} \qquad \boxed{\cosh\frac{\eta}{2} = \frac{\sqrt{|E|+mc^2}}{\sqrt{2mc^2}}} \qquad \boxed{\sinh\frac{\eta}{2} = \frac{\sqrt{|E|+mc^2}}{\sqrt{2mc^2}}\frac{pc}{|E|+mc^2}}$$

$$\Rightarrow S(\vec{p}) = \sqrt{\frac{|E|+mc^2}{2mc^2}}\left(\mathbb{1} - \begin{pmatrix} 0 & \frac{\vec{p}\vec{\sigma}}{p} \\ \frac{\vec{p}\vec{\sigma}}{p} & 0 \end{pmatrix}\frac{pc}{|E|+mc^2}\right)$$

$$\frac{\vec{p}\vec{\sigma}}{p} = \frac{1}{p}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \qquad \text{Def}: \; p_\pm = p_x \pm ip_y$$

$$\boxed{S(-\vec{p}) = \sqrt{\frac{|E|+mc^2}{2mc^2}}\begin{pmatrix} 1 & 0 & \frac{p_z c}{|E|+mc^2} & \frac{p_- c}{|E|+mc^2} \\ 0 & 1 & \frac{p_+ c}{|E|+mc^2} & \frac{-p_z c}{|E|+mc^2} \\ \frac{p_z c}{|E|+mc^2} & \frac{p_- c}{|E|+mc^2} & 1 & 0 \\ \frac{p_+ c}{|E|+mc^2} & \frac{-p_z c}{|E|+mc^2} & 0 & 1 \end{pmatrix}}$$

c) Raumspiegelung $\Lambda = P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$$

$$= \sum_{\nu=1}^{4} g^{\mu\nu}\gamma^\nu = g^{\mu\mu}\gamma^\mu \quad \text{ohne Summation}$$

- erfüllt durch $S = e^{i\chi}\cdot\gamma^0$ mit bel. χ z.B. $\chi = 0$

$$\Rightarrow \boxed{S^{-1}\gamma^\mu S = \gamma^0\gamma^\mu\gamma^0 = \begin{cases} \gamma^{0\,3} = \gamma^0 & \mu = 0 \\ -\gamma^{0\,2}\gamma^\mu = -\gamma^\mu & \mu \neq 0 \end{cases}}$$

- Raumspiegelungs-Operator $\mathcal{P}$

$$\psi(x) \longrightarrow \psi'(x') = \psi'(\vec{r}, t) = S\psi(x) = e^{i\chi}\cdot\gamma^0\psi(x) = e^{i\chi}\cdot\gamma^0\psi(-\vec{r}, t)$$

$$\boxed{\mathcal{P} = e^{i\chi}\cdot\gamma^0\, P_{orb}} \qquad P_{orb} \text{ macht } \vec{r} \to -\vec{r}$$